SYDNEY BOYS HIGH SCHOOL

4 UNIT MATHEMATICS

1997 TRIAL HSC EXAMINATION

Time allowed: 3 hours (plus 5 minutes Reading Time)

Total Marks: 120

Examiner: C. Kourtesis

INSTRUCTIONS:

- Attempt *all* questions.
- *All* questions are of equal value.
- All necessary working should be shown in every question. Full marks may not be awarded if work is careless or badly arranged.
- Standard integrals are provided. Approved calculators may be used.
- *Each* question attempted is to be returned in a *separate* Writing Booklet clearly marked Question 1, Question 2 etc. on the cover. Each booklet must show your name.
- If required, additional Writing Booklets may be obtained from the Examination Supervisor upon request.
- This is a trial paper and does not necessarily reflect the format or content of the HSC examination for this subject.

Question 1 (15 marks)

2 (a) Find:

(i) $$\int \frac{t^2 - 2}{t^3}\,dt$$

2 (ii) $$\int xe^x\,dx$$

3 (iii) $$\int \frac{2x\,dx}{(x+1)(x+3)}\,.$$

3 (b) By using the substitution $u = x - 4$ evaluate

$$\int_4^{4\cdot5} \frac{dx}{(x-3)(5-x)}$$

3 (c) (i) If $u_n = \int_0^{\pi/2} x^n \sin x\,dx, \quad n \geq 2$

prove that:

$$u_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)\,u_{n-2}$$

2 (ii) Hence evaluate

$$\int_0^{\pi/2} x^2 \sin x\,dx.$$

Question 2 (15 marks)

(a) The complex number w is given by $w = -1 + \sqrt{3}i$

2 (i) Show that $w^2 = 2\overline{w}$

2 (ii) Evaluate $|w|$ and arg w.

1 (iii) Show that w is a root of $w^3 - 8 = 0$

(b) Sketch the locus of z satisfying:

2 (i) $\mathrm{Re}(z) = |z|$

3 (ii) both $\mathrm{Im}(z) \geq 2$ and $|z - 1| \leq 2$

3 (c) Given that a and b are real numbers and

$$\frac{a}{1+i} + \frac{b}{1+2i} = 1$$

find the values of a and b..

3 (d) The complex numbers z_1 z_2, z_3 and z_4 are represented in the complex plane by the points A, B, C and D respectively.
Prove that:

If $z_1 + z_3 = z_2 + z_4$ then ABCD is a parallelogram.

Question 3 (15 marks)

(a) The equation $x^3 + bx^2 + x + 2 = 0$ where b is a real number has roots α, β, γ.

2 (i) Obtain an expression in terms of b for

$$\alpha^2 + \beta^2 + \gamma^2$$

1 (ii) Hence determine the set of possible values of b if the roots of the above equation are all real.

2 (iii) Write down the equation whose roots are

$$2\alpha, 2\beta, 2\gamma$$

3 (b) Given that the polynomial $P(x) = x^4 + x^3 - 3x^2 - 5x - 2$ has a zero of multiplicity 3, find all the zeros of P(x).

(c) If z represents a complex number such that

$$z^5 = 1 \text{ where } z \neq 1$$

2 (i) Deduce that $z^2 + z + 1 + \frac{1}{z} + \frac{1}{z^2} = 0$

2 (ii) By substituting $x = z + \frac{1}{z}$ reduce the equation in (i) to a quadratic in x.

3 (iii) Hence deduce that

$$\cos\frac{2\pi}{5}\cos\frac{4\pi}{5} = -\tfrac{1}{4}$$

Question 4 (15 marks)

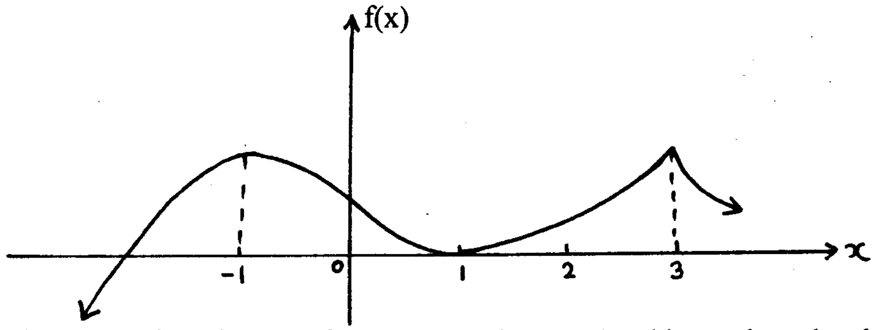
2 (a) (i) On the same set of axes sketch the graphs of

$$y = \tfrac{1}{2}x + 1 \quad \text{and} \quad y = |1 - x|.$$

2 (ii) Hence or otherwise determine the values of x for which

$$\tfrac{1}{2}x + 1 \geq |1 - x|$$

3 (b)



The graph of the function f(x) is drawn above. Use this graph to sketch the graph of $f'(x)$, the derivative of f(x).
You are given that there are turning points at x = -1 and x = 1.

3 (c) (i) Show that the curve $y = \dfrac{x^3 + 4}{x^2}$ has one stationary point but no point of inflexion and is always concave up.

2 (ii) Find the equations of any asymptotes.

1 (iii) Sketch the curve.

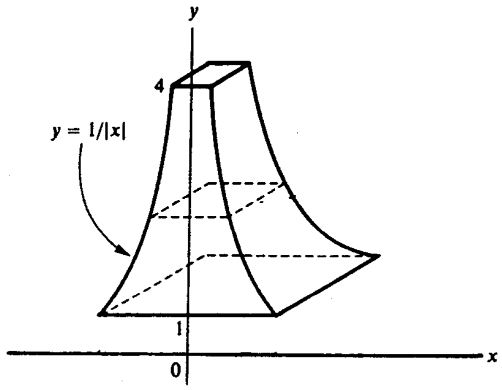
2 (iv) From the graph find the values of k for which the equation

$$x^3 - kx^2 + 4 = 0$$

has three distinct real roots.

Question 5 (15 marks)

5 0 (a) The plan of a steeple is bounded by the curve $y = \frac{1}{|x|}$ and the lines $y = 1$ and $y = 4$.



Each horizontal cross-section is a square.
Find the volume of the steeple.

1 (b) (i) Sketch the graph of $2y = y^2 - x + 1$

5 (ii) The region bounded by the graphs of $2y = y^2 - x + 1$ and $x = 1$ is rotated about the x axis.
By summing volumes of cylindrical shells show that the volume V of the resulting solid is

$$V = \frac{8\pi}{3} \text{ units}^3$$

4 (c) How many different ways are there of seating 4 married couples at a circular table with men and women in alternate positions and no wife next to her husband?

(Two seating arrangements are the same if each person has the same left and right hand neighbours).

Question 6 (15 marks)

(a) The tangent to the parabola $x^2 = 4ay$ at the point $P(2at, at^2)$ where t is the parameter, meets the directrix at Q. M is the midpoint of PQ.

2 (i) Show that the equation of the tangent at P is

$$x = \frac{y}{t} + at$$

3 (ii) Find the co ordinates of Q and M

4 (iii) Hence prove that as P moves on the parabola $x^2 = 4ay$ the locus of M is

$$x^2(2y + a) = a\,(3y + a)^2$$

(b) A diameter AB of a circle APB with centre 0 is produced to C so that AB = 2BC.

CT is the perpendicular from C to the tangent at P.

1 (i) Draw a clear diagram of the above

1 (ii) Explain why OP is parallel to CT.

4 (iii) Prove that BP = BT.

Question 8 (15 marks)

1 (a) (i) If a, b are positive numbers show that

$$\frac{a+b}{2} \geq \sqrt{ab}$$

2 (ii) Hence, or otherwise, if $a + b = 1$ prove that

$$a^2 + b^2 \geq \tfrac{1}{2}$$

2 (b) If $a > b > 0$ prove that

$$4b < (\sqrt{a} + \sqrt{b})^2 < 4a$$

3 (c) $P(x)$ is a polynomial of degree n with rational coefficients. If the leading coefficient is a_o and $\alpha_1\ \alpha_2\ \alpha_3, \ldots\ \alpha_n$ are the roots of $P(x) = 0$ prove that:

$$P^{1}(x) = \frac{P(x)}{(x-\alpha_1)} + \frac{P(x)}{(x-\alpha_2)} + \ldots\ldots\ldots + \frac{P(x)}{(x-\alpha_n)}$$

(d) If for $1 \leq x \leq n$, $f(x) \geq 0$ and $f''(x) < 0$, show, by comparing the area under the curve $y = f(x)$ between $x = 1$ and $x = n$ with the area of a region consisting of a suitably chosen sequence of trapezia, that

2 (i) $$\int_1^n f(x)\,dx > \sum_{r=2}^{n-1} f(r) + \tfrac{1}{2} f(1) + \tfrac{1}{2} f(n)$$

5 (ii) If $f(x) = \log_e x$ deduce that for n a positive integer

$$n! < n^{n+\frac{1}{2}} e^{1-n}$$

Question 7 (15 marks)

3 (a) (i) A projectile is fired, in a medium where air resistance is neglected, with an initial velocity V and an angle of projection θ. Prove that the equation of the trajectory referred to horizontal and vertical axes through the point of projection is

$$y = x \tan\theta - \frac{gx^2 \sec^2\theta}{2V^2}$$

where g is the acceleration due to gravity.

5 (ii) A projectile is fired horizontally with initial velocity V from a point O which is at the top of a cliff so as to hit a fixed target in the water and it is observed that the time of flight is T. The target can also be hit by firing a projectile with initial velocity V at an angle θ above the horizontal.
(Air resistance is neglected for both cases).
Show that the distance of the target from the point at sea-level vertically below O is given by

$$\frac{1}{2} g T^2 \tan\theta$$

(b) A particle of unit mass moves in a straight line under the action of a constant driving force F. It encounters a resistive force kv per unit mass where v is its speed and k is a positive constant.

(i) Write down the equation of motion.

3 (ii) Prove that if the speed increases from u to 2u over a time interval T, then

$$F = ku\left[\frac{2e^{kT} - 1}{e^{kT} - 1}\right]$$

3 (iii) Find the distance moved in this time interval

QUESTION 1:

a) i)
$$\int \frac{t^2-2}{t^3}\,dt = \int \frac{1}{t} - 2t^{-3}\,dt$$
$$= \ln|t| - \frac{2t^{-2}}{-2} + c$$
$$= \ln|t| + \frac{1}{t^2} + c$$

ii)
$$\int xe^x\,dx = \int x\,\frac{d}{dx}(e^x)\,dx$$
$$= xe^x - \int e^x\,dx \quad \text{(by parts)}$$
$$= xe^x - e^x + c$$

iii) Let
$$\int \frac{2x\,dx}{(x+1)(x+3)} = I = \int \frac{2x\,dx}{x^2+4x+3}$$

Let
$$\frac{1}{(x+1)(x+3)} = \frac{A}{x+1} + \frac{B}{x+3}$$
$$\therefore\ 1 = A(x+3) + B(x+1)$$

Set $x=-3$: $\quad 1 = B(-2) \Rightarrow B = -\frac{1}{2}$

Set $x=-1$: $\quad 1 = A(2) \Rightarrow A = \frac{1}{2}$

$$\therefore\ I = \int \frac{2x+4\ dx}{x^2+4x+3} - \int \frac{4\,dx}{(x+1)(x+3)}$$
$$= \ln|x^2+4x+3| - \frac{4}{2}\int \frac{1}{x+1} - \frac{1}{x+3}\,dx$$
$$= \ln|x^2+4x+3| - 2\ln|x+1| + 2\ln|x+3| + c$$
$$= \ln\left|\frac{(x+1)(x+3)(x+3)^2}{(x+1)^2}\right| + c$$
$$= \ln\left|\frac{(x+3)^3}{x+1}\right| + c$$

b) let $u = x-4$
$du = dx$

x	4	4.5
u	0	0.5

$$\therefore \int_4^{4.5} \frac{dx}{(x-3)(5-x)} = \int_0^{0.5} \frac{du}{(u+1)(1-u)}$$
$$= \frac{1}{2}\int_0^{0.5} \frac{1}{1-u} + \frac{1}{1+u}\,du$$
$$= \frac{1}{2}\Big[\ln|(1-u)(1+u)|\Big]_0^{0.5}$$
$$= \frac{1}{2}\left[\ln\left|\frac{\frac{1}{2}\cdot\frac{3}{2}}{1\cdot 1}\right|\right]$$
$$= \frac{1}{2}\ln\frac{3}{4}$$

c) i)
$$u_n = \int_0^{\pi/2} x^n \sin x\,dx \quad , n \geq 2$$
$$= \Big[-\cos x \,.\, x^n\Big]_0^{\pi/2} - \int_0^{\pi/2} n x^{n-1}(-\cos x)\,dx \quad \text{(by parts)}$$
$$= 0 + n\int_0^{\pi/2} x^{n-1}\cos x\,dx$$
$$= \Big[n x^{n-1}\sin x\Big]_0^{\pi/2} - n\int_0^{\pi/2}(n-1)x^{n-2}\sin x\,dx$$
$$= n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)u_{n-2}$$

// as required

ii)
$$u_2 = \int_0^{\pi/2} x^2 \sin x\,dx$$
$$u_0 = \int_0^{\pi/2} \sin x\,dx = \Big[-\cos x\Big]_0^{\pi/2} = 1$$
$$\therefore\ u_2 = 2\left(\frac{\pi}{2}\right)^1 - 2(1).1$$
$$= \pi - 2$$

PHOENIX MATHEMATICS
First Floor, 84-88 Bronte Road, Bondi Junction, 2022.
Ph: 9389-2699 Fax: 9386-1493
Web: www.maths.com.au Email: phoenix@maths.com.au

QUESTION 2:

a) i) $w = -1+\sqrt{3}i$

$w^2 = (-1+\sqrt{3}i)^2$

$= 1-3-2\sqrt{3}i$

$= -2-2\sqrt{3}i$

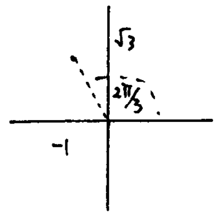
$2\bar{w} = 2(-1-\sqrt{3}i)$

$= w^2$, as required.

ii) $|w| = \sqrt{(-1)^2+(\sqrt{3})^2}$

$= 2$

$\arg w = \frac{2\pi}{3}$



iii) $w = 2\operatorname{cis}\frac{2\pi}{3}$

$\therefore LHS = w^3-8 = 8\operatorname{cis}3\left(\frac{2\pi}{3}\right)-8$

$= 8.1-8$

$= 0 = RHS$, as required

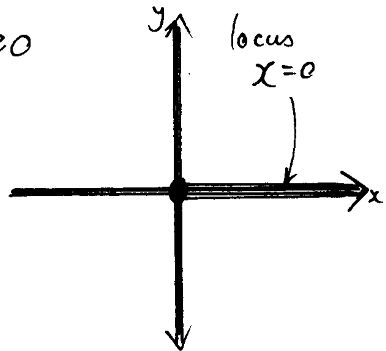
b) i) $Re(z) = |z|$, $\therefore x \geq 0$

$\therefore x = \sqrt{x^2+y^2}$

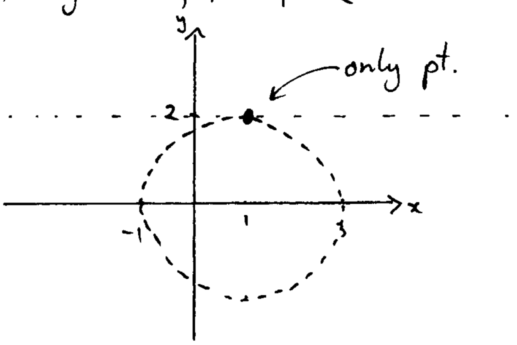
$x^2 = x^2+y^2$

$y^2=0$

$y=0,\ x\geq 0$



ii) $y \geq 2$, $|z-1| \leq 2$



c) $\frac{a}{1+i} + \frac{b}{1+2i} = 1$

$a+2ai+b+bi = (1+i)(1+2i)$

$(a+b)+(2a+b)i = 1-2+3i$

equating real and imaginary parts.

$a+b=-1$ ①

$2a+b = 3$ ②

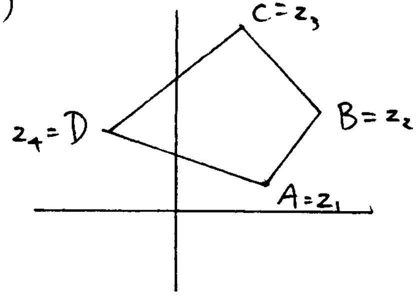
②−①: $a = 4$ ③

③⇒①: $4+b=-1$

$b=-5$

$\therefore \underline{a=4,\ b=-5}$

d)



Now $z_1+z_3=z_2+z_4$

$\therefore z_1-z_2=z_4-z_3$

ie $|z_1-z_2|=|z_4-z_3|$ ie $AB=CD$

$\arg(z_1-z_2)=\arg(z_4-z_3)$ $\therefore AB \parallel CD$

$\therefore$ since $AB=CD$ and $AB\parallel CD$

ABCD is a parallelogram. // as required

QUESTION 3

a) i) $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$= (-b)^2 - 2(1)$

$= b^2-2$

ii) all real if $\alpha^2+\beta^2+\gamma^2 \geq 0$

$\therefore b^2-2 \geq 0$

$\therefore -\sqrt{2} \leq b \leq \sqrt{2}$

iii) if $2\alpha, 2\beta, 2\gamma$ are roots of new equation y, then $\alpha = \frac{y}{2}$ is a root of the original equation.

$\therefore \left(\frac{y}{2}\right)^3 + b\left(\frac{y}{2}\right)^2 + \frac{y}{2} + 2 = 0$

$y^3 + 2by^2 + 4y + 16 = 0$

b) let α be the root of multiplicity 3.

$\therefore P(\alpha) = P'(\alpha) = P''(\alpha) = 0$

$P'(x) = 4x^3+3x^2-6x-5$

$P''(x) = 12x^2+6x-6$

let $P''(x)=0 \quad \therefore 2x^2+x-1 = 0$

$(2x-1)(x+1) = 0$

$\therefore x = \frac{1}{2}$, or -1

$P'(-1) = -4+3+6-5 = 0$

$\therefore -1$ is the triple root

since $P(x)$ is a quartic there are at most four roots. $\prod\alpha = -2$

$= (-1)^3\beta$, β is the other root

$\therefore \beta = 2$

$\therefore$ zeros of $P(x)$ are $-1, -1, -1, 2$.

c) i) $z^5-1=0$

$\therefore (z-1)(z^4+z^3+z^2+z+1) = 0$

but $z \neq 1 \quad \therefore z^4+z^3+z^2+z+1 = 0$

but $z \neq 0 \quad \therefore z^2+z+1+\frac{1}{z}+\frac{1}{z^2} = 0$

ii) let $x = z+\frac{1}{z}$

$\therefore x^2 = z^2+2+\frac{1}{z^2}$

$\therefore$ equation in (i) becomes

$x^2+x-1=0$

iii) now $x = z+\frac{1}{z}$, but $|z|=1$

$\therefore x = z+\bar{z}$

$= 2\,\mathrm{Re}(z)$

from (i) $z = \operatorname{cis}\frac{\pm 2\pi}{5}$ or $\operatorname{cis}\frac{\pm 4\pi}{5}$.

$\therefore x = 2\cos\frac{2\pi}{5}$ or $2\cos\frac{4\pi}{5}$

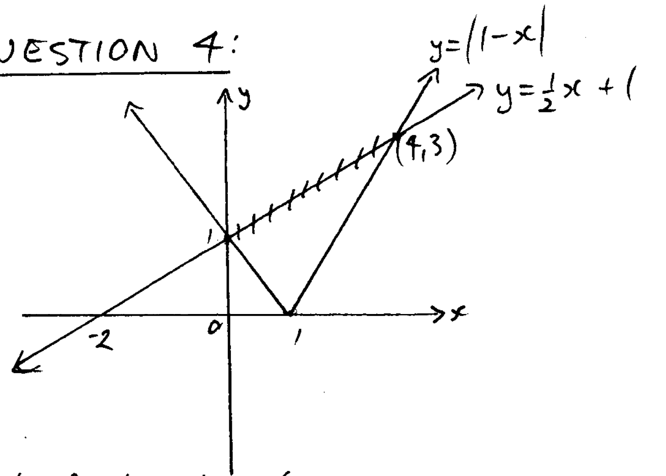
$\therefore$ product of roots from eqn. in (ii) gives

$2\cos\frac{2\pi}{5} \cdot 2\cos\frac{4\pi}{5} = -1$

$\therefore \cos\frac{2\pi}{5}\cos\frac{4\pi}{5} = -\frac{1}{4}$ // as required.

QUESTION 4:

a)(i)



ii) pts of intersection $(0,1)$,

$x-1 = \frac{1}{2}x+1$

$\frac{1}{2}x = 2$

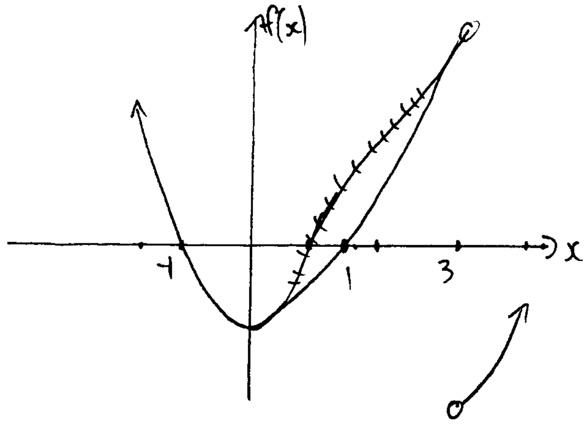
$x = 4$

$\therefore (0,1), (4,3)$.

$\therefore \frac{1}{2}x+1 \geq |1-x|$

for $\underline{0 \leq x \leq 4}$.

)



c)(i) $y = \dfrac{x^3+4}{x^2} = x + 4x^{-2}$

$y' = 1 - \dfrac{2.4}{x^3}$

$= 1 - \dfrac{8}{x^3}$

Let $y'=0$ for stationary point.

$\therefore 0 = 1 - \dfrac{8}{x^3}$

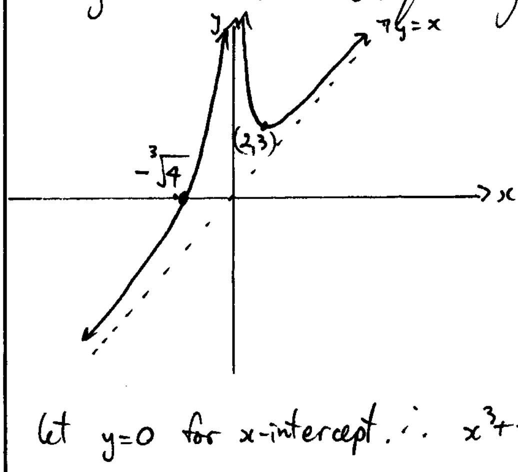
$\therefore x = 2, y = 3$

$y'' = \dfrac{24}{x^4} > 0$ for all x

$\therefore$ there is no point of inflexion, it is always concave up and $(2,3)$ is a minimum turning point.

ii) $x=0$ is a vertical asymptote

$y=x$ is an oblique asymptote.



Let $y=0$ for x-intercept. $\therefore x^3+4=0$

$x = -\sqrt[3]{4}$

iv) $x^3-kx^2+4=0$

$\dfrac{x^3+4}{x^2} = k$

ie ~~for~~ what horizontal lines pass through the graph 3 times. ie $\underline{k>3}$.

QUESTION 5:

a) Volume of a typical slice,

$$\delta V = (2x)^2 \delta y = 4x^2 \delta y$$

$$\therefore V \doteqdot \sum_{y=1}^{4} \delta V$$

$$= \sum_{y=1}^{4} 4x^2 \delta y$$

$$\therefore V = \lim_{\delta y \to 0} \sum_{y=1}^{4} 4 . \frac{1}{y^2} \delta y$$

$$= \int_1^4 \frac{4}{y^2} dy$$

$$= \left[\frac{-4}{y} \right]_1^4$$

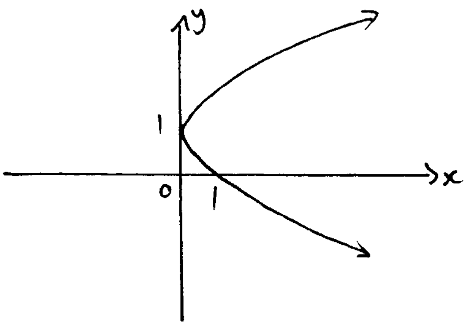
$$= 4\left(1 - \frac{1}{4}\right)$$

$$= 3 \quad u^3$$

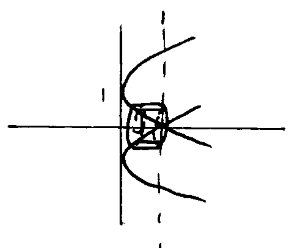
b) (i) $2y = y^2 - x + 1$

$x = y^2 - 2x + 1$

$x = (y-1)^2$



(ii)



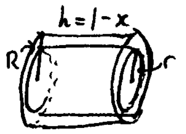
$\pm\sqrt{x} = y - 1$

$y = 1 \pm \sqrt{x}$

$h = 1 - x = 2y - y^2$

$r = y$

$R = y + \delta y$



typical shell volume $\delta V = \pi(R^2 - r^2)h$

$$\doteqdot \pi\left((y+\delta y)^2 - y^2\right)(1-x)$$

$$= \pi(y + \delta y + y)(y + \delta y - y)(1-x)$$

$$= \pi(2y + \delta y)\delta y(1-x)$$

$$= \pi(1-x)(2y\,\delta y + \delta y^2)$$

$$\doteqdot \pi(1-x)\,2y\,\delta y,$$

since δy^2 is negligible.

Now $V = \lim_{\delta y \to 0} \sum_{y=0}^{2} \pi(1-x)2y\,\delta y$

$$= \int_0^2 \pi(2y - y^2)2y\,dy$$

$$= 2\pi \int_0^2 2y^2 - y^3\,dy$$

$$= 2\pi\left[\frac{2}{3}y^3 - \frac{1}{4}y^4\right]_0^2$$

$$= 2\pi\left[\frac{16}{3} - 4 - 0\right]$$

$$= 2\pi . \frac{4}{3}$$

$$= \frac{8\pi}{3}$$

// as required.

c) fix one combo of 4 men, only 2 places for W_1, choose 1. Then only 1 place for remaing women.

$\therefore$ No. of combos $= \underbrace{3!}_{\text{orderings of men.}} \times 2$

$= 12$

QUESTION 6:

a) i) $\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{2at}{2a} = t$

$\therefore$ equation of tgt at P is

$$y - at^2 = t(x - 2at)$$

$$\frac{y}{t} - at = x - 2at$$

$$\frac{y}{t} + at = x \quad \text{// as required.}$$

ii) directrix is $y = -a$

$\therefore$ for Q, let $y = -a$

$$\therefore \frac{-a}{t} + at = x$$

$$\therefore Q \text{ is } \left(at - \frac{a}{t}, -a\right)$$

$$M = \left(\frac{at - \frac{a}{t} + 2at}{2}, \frac{-a + at^2}{2}\right)$$

$$= \left(\frac{3at}{2} - \frac{a}{2t}, \frac{at^2}{2} - \frac{a}{2}\right)$$

iii) $LHS = x^2(2y + a)$

$$= \left(\frac{3at}{2} - \frac{a}{2t}\right)^2 (at^2 - a + a)$$

$$= at^2 \cdot \frac{a^2}{4}\left(3t - \frac{1}{t}\right)^2$$

$$= \frac{a^3t^2}{4}\left(9t^2 - 6a + \frac{a^2}{t^2}\right)$$

$RHS = a(3y + a)^2$

$$= a\left(\frac{3at^2}{2} - \frac{3a}{2} + a\right)^2$$

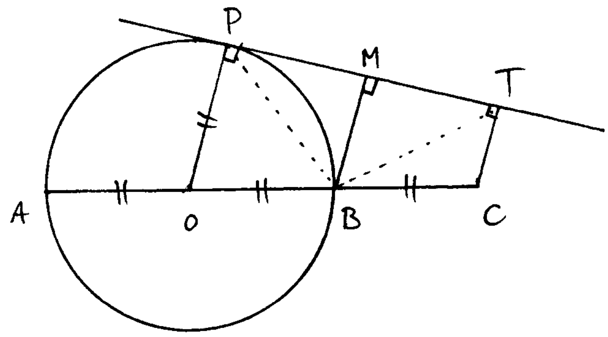
$$= a^3\left(\frac{3t^2}{2} - \frac{1}{2}\right)^2$$

$$= \frac{a^3}{4}\left(9t^4 - 6t^2 + 1\right)$$

$$= \frac{a^3t^2}{4}\left(9t^2 - 6 + \frac{1}{t^2}\right)$$

$= LHS$ // as required

b) (i)



(ii) $\angle OPT + \angle OCT = 90° + 90°$ (angle made by tgt and radius)

$= 180°$

$\therefore OP \parallel TC$ (converse of cointerior angles are supplementary)

iii) construct perpendicular from B to PT at M. Similarly to (ii) $OP \parallel BM \parallel CT$.

$\therefore \frac{MT}{PM} = \frac{BC}{BO} = \frac{1}{1}$ (parallel lines cut lines in same proportions)

$\therefore MT = PM$

also in Δs PMB + TMB,

MB is common,

$\angle PMB = \angle TMB = 90°$

$\therefore \Delta PMB \equiv \Delta TMB$ (SAS)

$\therefore BP = BT$ (corresponding sides of congruent triangles)

QED.

QUESTION 7

a)(i) $\ddot{x}=0 \qquad \ddot{y}=-g$

$\dot{x}=V\cos\theta \qquad \dot{y}=-gt+V\sin\theta$

$x=Vt\cos\theta+c \qquad y=\frac{-gt^2}{2}+Vt\sin\theta+d$

at $t=0$, $(x,y)=(0,0)$ $\therefore c=d=0$

$\therefore t=\frac{x}{V\cos\theta}$

$\therefore y=\frac{-g}{2}\left(\frac{x}{V\cos\theta}\right)^2+V\left(\frac{x}{V\cos\theta}\right)\sin\theta$

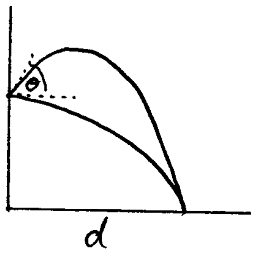
$=-\frac{gx^2\sec^2\theta}{2V^2}+x\tan\theta$

// as required

ii) when $\theta=0$, $t=T$, let distance from point to target be d.

$\therefore T=\frac{d}{V\cos0}$

$\therefore V=\frac{d}{T}$



$y_{\theta=0}=\frac{-gx^2}{2V^2}$

$y_{\theta=\theta}=x\tan\theta-\frac{gx^2\sec^2\theta}{2V^2}$

Now d satisfies when $y_{\theta=0}=y_{\theta=\theta}$

$\therefore \frac{-gd^2}{2V^2}=d\tan\theta-\frac{gd^2\sec^2\theta}{2V^2}$

$0=d\tan\theta+\frac{gd^2}{2V^2}(1-\sec^2\theta)$

$=d\tan\theta-\frac{gd^2\tan^2\theta}{2V^2}$

$=d\tan\theta\left(1-\frac{gd\tan\theta}{2V^2}\right)$

$\therefore d=0$ or $d=\frac{2V^2}{g\tan\theta}$

but $d>0$

$\therefore d=\frac{2\left(\frac{d}{T}\right)^2}{g\tan\theta}$

$\therefore d=\frac{1}{2}gT^2\tan\theta$

// as required.

b)i) $m=1$ $\therefore \ddot{x}=F-kv$

ii) let $t=\alpha$ at beginning of time interval.

$\frac{dv}{dt}=F-kv$

$\frac{dt}{dv}=\frac{1}{F-kv}$

$\int_\alpha^{\alpha+T}dt=\int_u^{2u}\frac{1}{F-kv}\,dv$

$\left[t\right]_\alpha^{\alpha+T}=-\frac{1}{k}\left[\ln(F-kv)\right]_u^{2u}$

$T=-\frac{1}{k}\ln\left|\frac{F-2ku}{F-ku}\right|$

$e^{-kT}=\frac{F-2ku}{F-ku}$

$F-ku=Fe^{kT}-2kue^{kT}$

$F(e^{kT}-1)=2kue^{kT}-ku$

$F=\frac{ku(2e^{kT}-1)}{e^{kT}-1}$

// as required

7. b) iii) $x = \int_{\alpha}^{\alpha+T} v \, dt$

$$\ddot{x} = F - kv$$

$$\frac{dv}{dt} = F - kv$$

$$t = \int \frac{1}{F-kv} dv$$

$$= -\frac{1}{k} \ln|F-kv| + c$$

$$\alpha = -\frac{1}{k} \ln|F-ku| + c$$

$$c = \alpha + \frac{1}{k} \ln|F-ku|$$

$$\therefore t = \alpha + \frac{1}{k} \ln\left(\frac{F-ku}{F-kv}\right)$$

$$e^{k(t-\alpha)} = \frac{F-ku}{F-kv}$$

$$F-kv = (F-ku)e^{-k(t-\alpha)}$$

$$v = \frac{1}{k}\left(F - \frac{(F-ku)e^{-k(t-\alpha)}}{\;}\right)$$

$$x = \int_{\alpha}^{\alpha+T} v \, dt$$

$$= \frac{1}{k}\left[\left(Ft + \frac{(F-ku)}{k} e^{-k(t-\alpha)}\right)\right]_{\alpha}^{\alpha+T}$$

$$= \frac{1}{k}\left(FT + \frac{F-ku}{k}\left(e^{-kT} - 1\right)\right)$$

$$= \frac{1}{k}\left(FT + ku\left(\frac{2e^{kT} - 1 - e^{kT} + 1}{k(e^{kT}-1)} \cdot \frac{(1-e^{kT})}{e^{kT}}\right)\right)$$

$$= \frac{1}{k}\left(FT + \frac{u(-e^{kT})}{e^{kT}}\right)$$

$$= \frac{1}{k}(FT - u)$$

$$= \frac{1}{k}\left(Tku\left[\frac{2e^{kT}-1}{e^{kT}-1}\right] - u\right)$$

QUESTION 8.

1 (a) $(a-b)^2 \geqslant 0$

$a^2+b^2-2ab \geqslant 0$

$\dfrac{a^2+b^2}{2} \geqslant ab$

substitute a for a^2, b for b^2

$\therefore \dfrac{a+b}{2} \geqslant \sqrt{ab}$ // as required.

(ii) $(a+b)^2 = a^2+b^2+2ab$

but $a+b=1$ $\therefore$ $a^2+b^2 = 1-2ab$

but $ab \leq \left(\dfrac{a+b}{2}\right)^2 = \dfrac{1}{4}$

$\therefore a^2+b^2 \geqslant 1-2\left(\frac{1}{4}\right)$

$= \frac{1}{2}$ // as required

b) $(\sqrt{a}+\sqrt{b})^2 = a+b+2\sqrt{ab}$

$\leq a+a+2\sqrt{aa}$

$=4a$

also $a+b+2\sqrt{ab} \geqslant b+b+2\sqrt{b.b}$

$=4b$

result follows.

c) $P(x) = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\dots(x-\alpha_n)$

$P'(x) = 1.(x-\alpha_2)(x-\alpha_3)\dots(x-\alpha_n) +$

$(x-\alpha_1).1.(x-\alpha_3)\dots(x-\alpha_n) +$

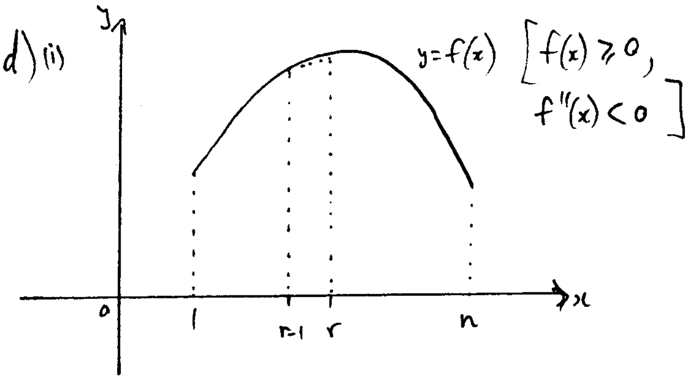
$\vdots$

$(x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_{n-1}).1.$

$= \dfrac{P(x)}{x-\alpha_1} + \dfrac{P(x)}{x-\alpha_2} + \dots + \dfrac{P(x)}{x-\alpha_n}$

// as required.

d) (i)



Because $f(x) \geqslant 0$ and $f''(x)<0$, the curve always lies above the trapezium.
Now area of trapezium from $x=r-1$ to $x=r$

$= \frac{1}{2}\left(f(r-1)+f(r)\right).\left(r-(r-1)\right)$

$= \frac{1}{2}\left(f(r-1)+f(r)\right)$

Because the curve always lies above the trapezia the area under the curve is greater than the area of the trapezia.

$\therefore \displaystyle\int_1^n f(x)\,dx > \sum_{r=2}^{n}\left(\tfrac{1}{2}\left(f(r-1)+f(r)\right)\right)$

$= \left[\frac{1}{2}f(1)+\frac{1}{2}f(2)\right]+\left[\frac{1}{2}f(2)+\frac{1}{2}f(3)\right]+\dots$

$+\left[\frac{1}{2}f(n-1)+\frac{1}{2}f(n)\right]$

$= \frac{1}{2}f(1)+\frac{1}{2}f(n)+f(2)+f(3)+\dots+f(n-1)$

$= \displaystyle\sum_{r=2}^{n-1} f(r) + \frac{1}{2}f(1)+\frac{1}{2}f(n)$

// as required.

(ii) $f(x) = \log_e x \geqslant 0$ for $x \geqslant 1$.

$f'(x) = \dfrac{1}{x}$

$f''(x) = \dfrac{-1}{x^2} < 0$ for $x>1$

$\therefore$ from part (i)

$\displaystyle\int_1^n \log_e x\,dx > \sum_{r=2}^{n-1}\log_e r + \frac{1}{2}\log_e 1 + \frac{1}{2}\log_e n.$